كلية الهندسة بشبرا
FACULTY OF ENGINEERING AT SHOUBRA

Electrical & Magnetic Fields
Electromagnetic Fields / Fundamentals
(ELE222)(ELE242)(CCE302)

Lecture (05)

Electric Field Intensity (Line and Surface Charge Density)

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Electric Field Due to Continuous Charge Distributions

- So far, we have only considered Forces and Fields due to point charges. It is also possible to have continuous charge distribution along a line or a surface or in a volume.
- It is customary to denote the line charge density, ρ_L [C/m], surface charge density , ρ_S [C/m^2], and volume charge density, ρ_v [C/m^3] as shown in Fig. 2.9.
- The electric field due to each of the charge distributions ρ_L, ρ_S and ρ_v may be regarded as the summation of field contributed by the numerous point charges making up charge distributions.
- Thus the electric field intensity for these continuous charge distribution systems can be found by modifying the eqn. (2.9) to become:

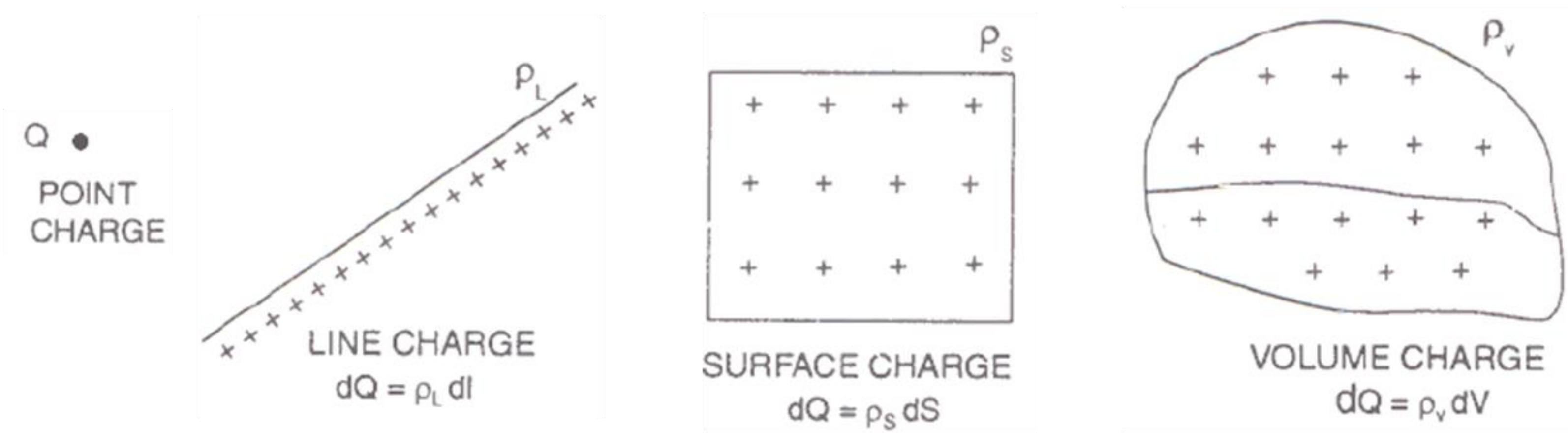


Fig. 2.9 Different types of charge distributions.

Electric Field Due to Continuous Charge Distributions

$$\vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{a}_r = \int \frac{dQ}{4\pi\varepsilon_o r^2}\hat{a}_r \; V/m$$

- Electric field intensity, $\vec{E}$ can be found by replacing dQ in (2.12) with differential charge element as: [$\rho_L dl$] for line charge density, [$\rho_S ds$] for surface charge density and $\rho_v dv$ for volume charge density, and then integrating.
- For line charge distributions, the electric field intensity $\vec{E}$ is given by:

$$\vec{E} = \int_L \frac{\rho_L dl}{4\pi\varepsilon_o r^2}\hat{a}_r \; V/m$$

- For surface charge distribution, the electric field intensity $\vec{E}$ is given by:

$$\vec{E} = \int_S \frac{\rho_s dS}{4\pi\varepsilon_o r^2}\hat{a}_r \; V/m$$

- For volume charge distribution, the electric field intensity $\vec{E}$ is given by:

$$\vec{E} = \int_v \frac{\rho_v dv}{4\pi\varepsilon_o r^2}\hat{a}_r \; V/m$$

1 - Line Charge (Finite – infinite length)

Case (1) Find the components of the electric field intensity at point $P(r, 0, 0)$ for a finite line charge along the z axis from z_1 to z_2 and charged uniformly with a charge density of ρ_ℓ c/m.

⇒ Solution :

In cylindrical co. (r, ϕ, z) :

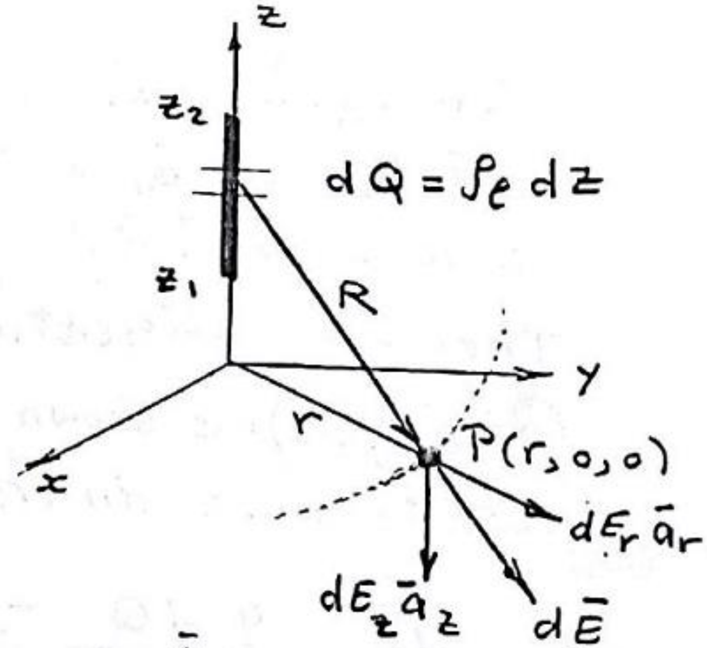
$$\bar{R} = r\,\bar{a}_r - z\,\bar{a}_z \qquad \text{as shown in fig. 3-7}$$

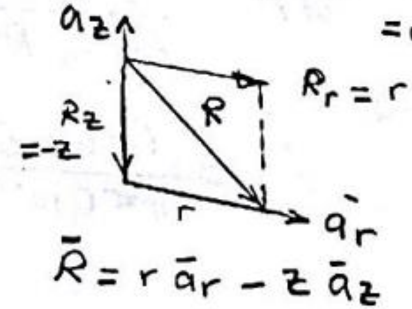
$$\therefore R = \sqrt{r^2 + z^2}$$

Then the differential charge ($dQ = \rho_\ell\, d\ell$) results in a differential electric field E :

$$d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{\rho_\ell\, dz}{4\pi\epsilon_0 (r^2+z^2)} \cdot \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}$$





Case (1)

$$\therefore \bar{E} = \frac{\rho_\ell}{4\pi\epsilon_0}\int_{z_1}^{z_2}\frac{r\,dz}{(r^2+z^2)^{3/2}}\bar{a}_r - \int_{z_1}^{z_2}\frac{z\,dz}{(r^2+z^2)^{3/2}}\bar{a}_z \qquad \bar{E} = \bar{E}_r\bar{a}_r + \bar{E}_z\bar{a}_z$$

$$= \underbrace{\frac{\rho_\ell r}{4\pi\epsilon_0}\left[\frac{z}{r^2\sqrt{r^2+z^2}}\right]_{z_1}^{z_2}}_{E_r}\bar{a}_r - \underbrace{\frac{\rho_\ell}{4\pi\epsilon_0}\left[\frac{-1}{\sqrt{r^2+z^2}}\right]_{z_1}^{z_2}}_{E_z}\bar{a}_z$$

$$\bar{E}_r = \frac{\rho_\ell}{4\pi\epsilon_0 r}\left[\frac{z_2}{\sqrt{r^2+z_2^2}} - \frac{z_1}{\sqrt{r^2+z_1^2}}\right]\bar{a}_r$$

$$\bar{E}_z = \frac{\rho_\ell}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+z_2^2}} - \frac{1}{\sqrt{r^2+z_1^2}}\right]\bar{a}_z$$

$$\bar{E} = E\,\bar{a}_R = E_r\,\bar{a}_r + E_z\,a_z$$

$$\Rightarrow \therefore \bar{E} = \bar{E}_r + \bar{E}_z$$

Case (2) ***Infinite Line Charge***

$$d\bar{E} = \frac{dQ}{4\pi\epsilon R^2}\,\bar{a}_R = \frac{\rho_l\,dl}{4\pi\epsilon R^2}\,\bar{a}_R$$

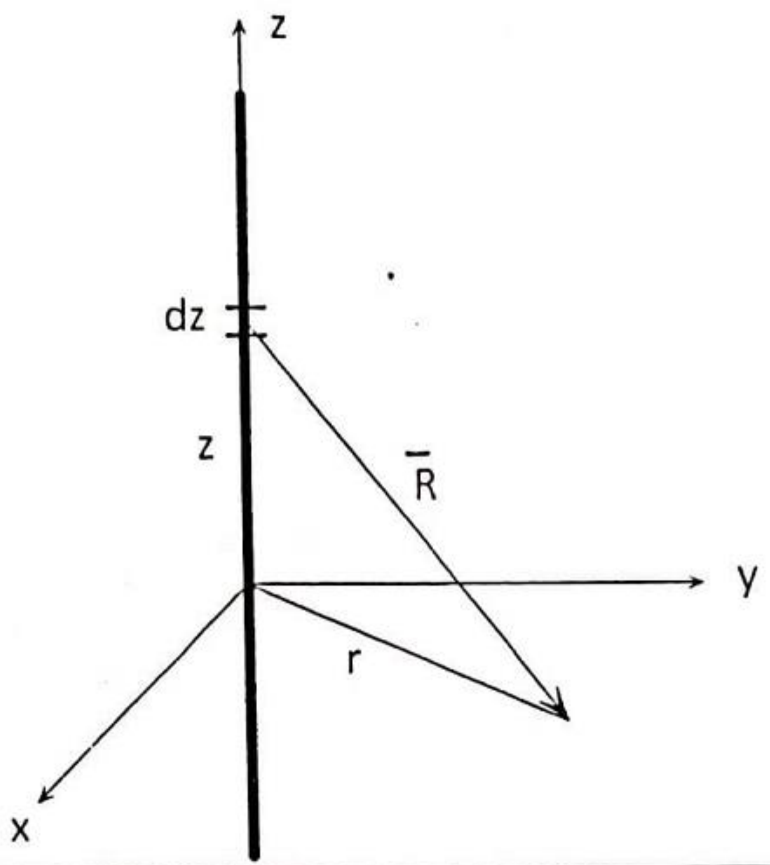
The total field at P is

$$\bar{E} = \int d\bar{E} = \int_L \frac{\rho_l\,dl}{4\pi\epsilon R^2}\,\bar{a}_R = \frac{\rho_l}{4\pi\epsilon}\int_L \frac{dl}{R^2}\,\bar{a}_R$$

$$\bar{R} = z(-\bar{a}_z) + r(\bar{a}_r)$$

$$R = |\bar{R}| = \sqrt{z^2 + r^2}$$

$$\bar{a}_R = \frac{z(-\bar{a}_z) + r(\bar{a}_r)}{\sqrt{z^2 + r^2}}$$



Case (2) *Infinite Line Charge*

$$\bar{E} = \frac{\rho_l}{4\pi\epsilon} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + r^2)} \left[\frac{r\,\bar{a}_r + z\,(-\bar{a}_z)}{\sqrt{z^2 + r^2}}\right]$$

Due to line symmetry , There is a radial component only

$$\bar{E} = \frac{\rho_l}{4\pi\epsilon} \int_{-\infty}^{\infty} \frac{r\,dz\,\bar{a}_r}{(z^2 + r^2)^{\frac{3}{2}}}$$

$$\bar{E} = \frac{\rho_l\,r}{4\pi\epsilon} \left[\frac{z}{r^2\sqrt{z^2 + r^2}}\right]_{-\infty}^{\infty} \bar{a}_r$$

$$\bar{E} = \frac{\rho_l}{4\pi\epsilon\,r} [1 - (-1)]\bar{a}_r$$

$$\boxed{\bar{E} = \frac{\rho_l}{2\pi\epsilon\,r}\,\bar{a}_r}$$

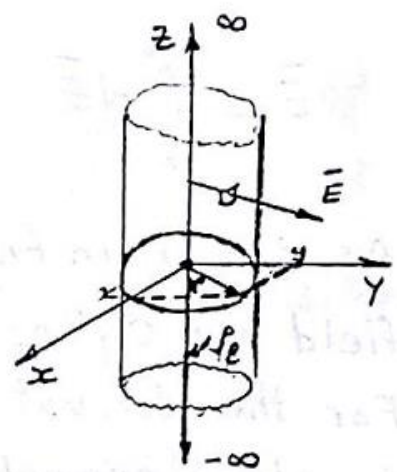
$\bar{E}$: المجال الكهربى عند نقطة بسبب خط طوله ∞ مشحون بـ ρ_l

r : البعد العمودى بين الخط و النقطة

$\bar{a}_r$: متجه وحدة من الخط للنقطة

Ex (1)

A uniform line charge, infinite in extent, with $\rho_\ell = 20$ nC/m, lies along the z axis. Find the electric field intensity at (6, 8, 3) m.



⇒ Solution :

It is required to find $\bar{E}$ at P = (x, y, z) = (6, 8, 3) m.

In cylindrical coordinates :

$$r = \sqrt{x^2 + y^2} \quad \text{(see chapter 1)}$$

$$\therefore r = \sqrt{6^2 + 8^2} = 10 \text{ m.}$$

The field is constant with z. Thus from the above derivation we have :

$$\bar{E} = \frac{\rho_\ell}{2\pi \epsilon_0 r} \bar{a}_r$$

$$= \frac{20 \times 10^{-9}}{2\pi (10^{-9}/36\pi) 10} \bar{a}_r \simeq 36 \bar{a}_r \text{ V/m}$$

2 – Surface Charge

Case2. Electric Field Intensity for Surface Charge Distribution

the contribution to E_x at P from this differential-width strip is then:

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o \rho}\hat{a}_\rho = \frac{\rho_L \vec{R}_P}{2\pi\varepsilon_o |\vec{R}_P|^2} \; V/m \;\Rightarrow\; dE_x = \frac{\rho_s dy}{2\pi\varepsilon_o \sqrt{x^2+y^2}}\cos\theta = \frac{\rho_s}{2\pi\varepsilon_o}\frac{x dy}{x^2+y^2}$$

Adding the effects of all the strips, we get:

$$E_x = \frac{\rho_s}{2\pi\varepsilon_o}\int \frac{x dy}{x^2+y^2} = \frac{\rho_s}{2\pi\varepsilon_o}\tan^{-1}\frac{y}{x}\Big|$$

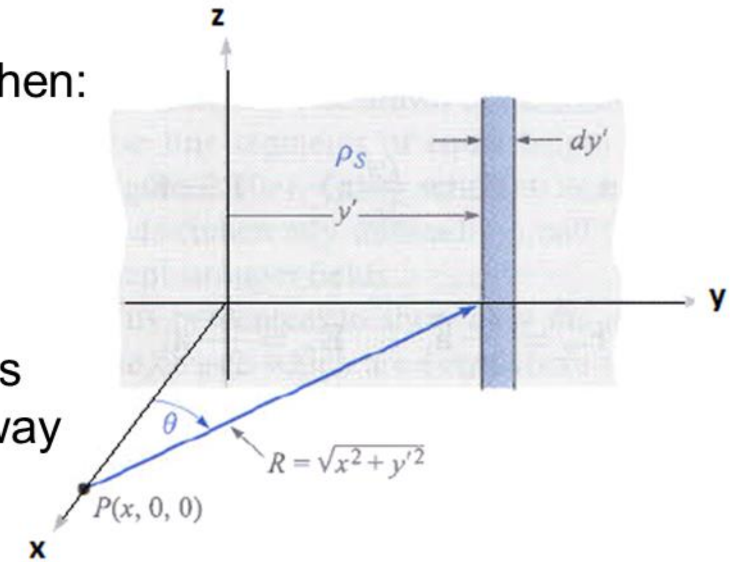
For infinite sheet $y \to \infty$ and Ex is given by:

$$E_x = \frac{\rho_s}{2\pi\varepsilon_o}\tan^{-1}\frac{y}{x}\Big|_{-\infty}^{\infty} = \frac{\rho_s}{2\varepsilon_o}$$

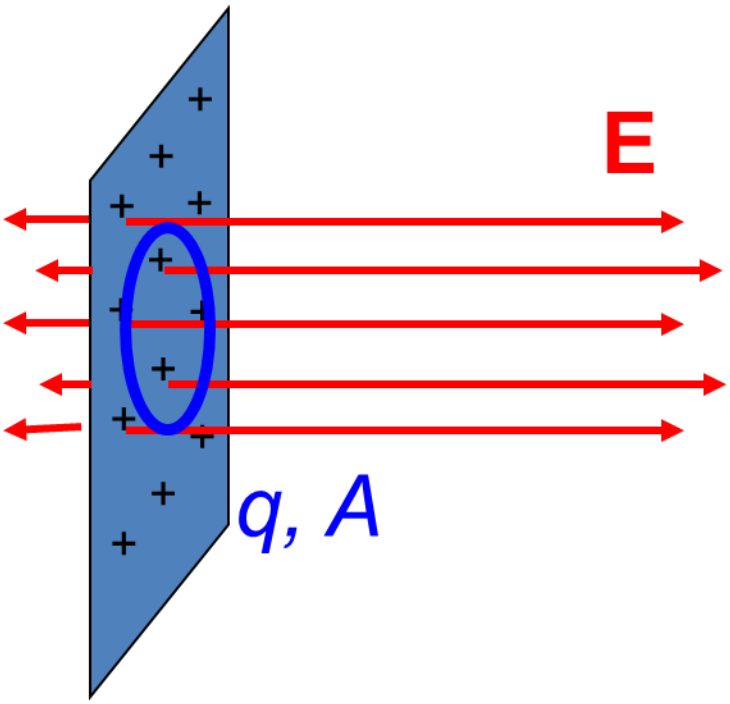
If the point P were chosen on the negative x axis, then:

$$E_x = -\frac{\rho_s}{2\varepsilon_o} \qquad (2.25c)$$

- For the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector $\vec{a}_n$, which is normal to the sheet and directed outward, or away from it. Then: $\vec{E} = \frac{\rho_s}{2\varepsilon_o}\hat{a}_n$ (2.25d)



➢ *So The Electric field of infinite plane of charge:*



Field is uniform and constant to ∞, in both directions

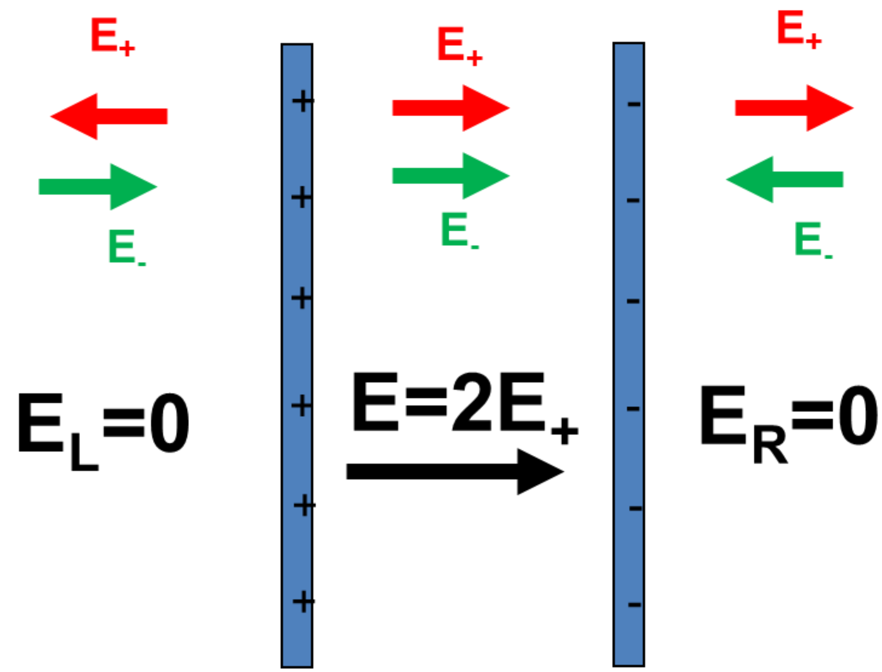
Electric field is proportional to the line density, and therefore to the charge density, $\boldsymbol{\rho_S = q/A}$

$$\vec{E} = \frac{\rho_S}{2\varepsilon_0}\hat{a}_N$$

Ex (3)

➢ ***If a second infinite sheet of charge, having a negative charge density $-\rho_S$ is located in the plane $x = a$, we may find the total field by adding the contribution of each sheet (Parallel plate capacitor)***

(assume separation small compared to the size)



$$E = 0 \qquad E = \rho_S / \varepsilon_0 \qquad E = 0$$

In the region $x > a$:

$$E_+ = \frac{\rho_S}{2\varepsilon_o} a_x \qquad E_- = -\frac{\rho_S}{2\varepsilon_o} a_x \qquad E = E_+ + E_- = 0$$

For $x < 0$:

$$E_+ = -\frac{\rho_S}{2\varepsilon_o} a_x \qquad E_- = \frac{\rho_S}{2\varepsilon_o} a_x \qquad E = E_+ + E_- = 0$$

For $0 < x < a$:

$$E_+ = \frac{\rho_S}{2\varepsilon_o} a_x \qquad E_- = \frac{\rho_S}{2\varepsilon_o} a_x \qquad E = E_+ + E_- = \frac{\rho_S}{\varepsilon_o} a_x$$

Case (4) Electric Field Intensity for A circular disk Charge Distribution

where $$d\vec{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\hat{a}_R = \frac{\rho_s r dr d\phi}{4\pi\varepsilon_o (r^2 + h^2)}\hat{a}_R$$

The vertical component of $\vec{E}$ along (or) parallel to Z-axis is given by :

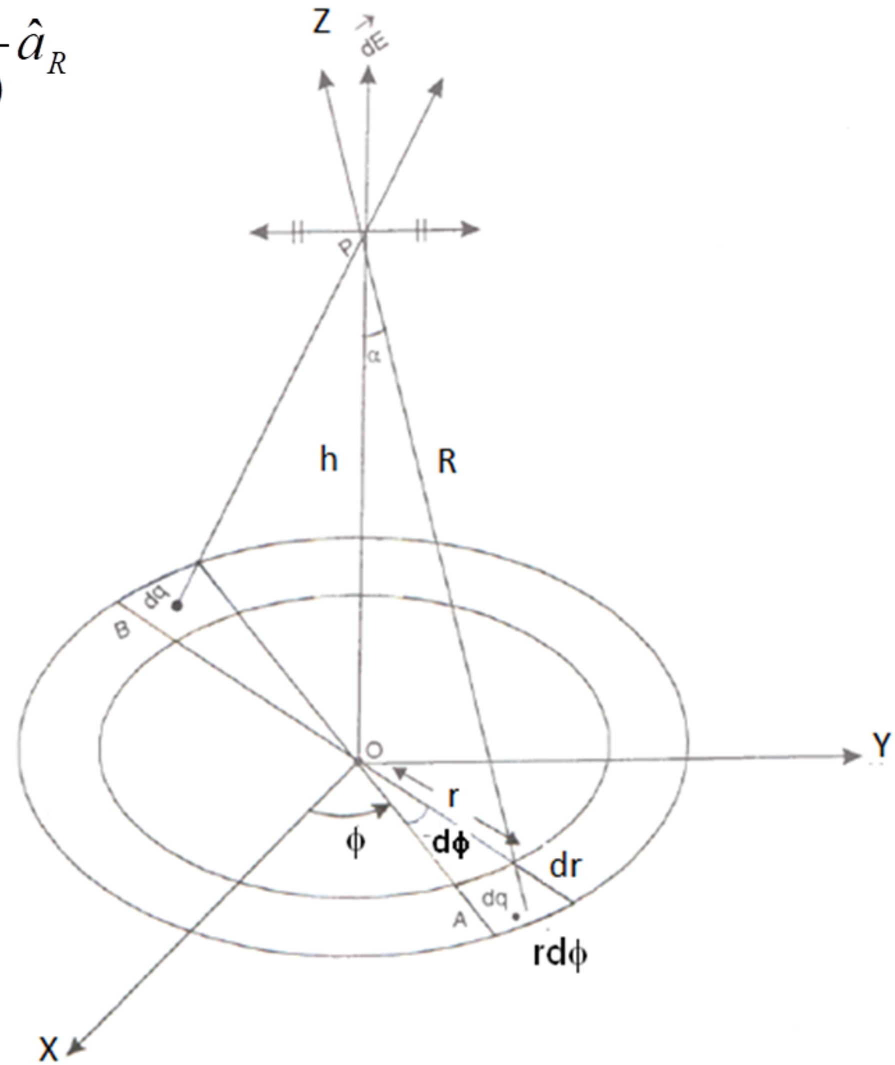
$$d\vec{E} = \frac{\rho_s r dr d\phi}{4\pi\varepsilon_o (r^2 + z^2)}\cos\alpha\ \hat{a}_z$$

so

$$\vec{E} = \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=2\pi}\frac{\rho_s r dr d\phi}{4\pi\varepsilon_o (r^2 + h^2)}.\cos\alpha\ \hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon_o}\int_{r=0}^{r=a}\frac{\cos\alpha}{(r^2 + h^2)}rdr\int_{\phi=0}^{\phi=2\pi}d\phi\ \hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_o}\int_{r=0}^{r=a}\frac{\cos\alpha}{(r^2 + h^2)}rdr\ \hat{a}_z$$



Electric Field Due to Continuous Charge Distributions

Electric Field Intensity for A circular disk Charge Distribution

From Fig. 2.13, we get: $r = h\tan\alpha \;\Rightarrow\; dr = h\sec^2\alpha$ and $r^2 + h^2 = h^2\sec^2\alpha$

Substitute these vales in the above equation, electric field intensity $\vec{E}$ is:

$$\vec{E} = \frac{\rho_s}{2\varepsilon_o}\int_{\alpha=0}^{\alpha}\frac{(h\tan\alpha)(h\sec^2\alpha)}{h^2\sec^2\alpha}\cos\alpha d\alpha\;\hat{a}_z = \frac{\rho_s}{2\varepsilon_o}\int_{\alpha=0}^{\alpha}\sin\alpha d\alpha\;\hat{a}_z$$

$$= \frac{\rho_s}{2\varepsilon_o}[-\cos\alpha]_0^{\alpha}\;\hat{a}_z = \frac{\rho_s}{2\varepsilon_o}(1-\cos\alpha)\;\hat{a}_z$$

If the radius, a becomes infinite the $\alpha \rightarrow 90^{o}$, we get:

$$\vec{E} = \frac{\rho_s}{2\varepsilon_o}\;\hat{a}_z$$

EX(4) *E-Field due to circular disk of raduis 'a' charged uniformly with surface charge density 'ρs'*

Find the force on a point charge of 50 μC at (0, 0, 5) m due to a charge of 500π μC that is uniformly distributed over the circular disk r ≤ 5 m, z = 0 m.

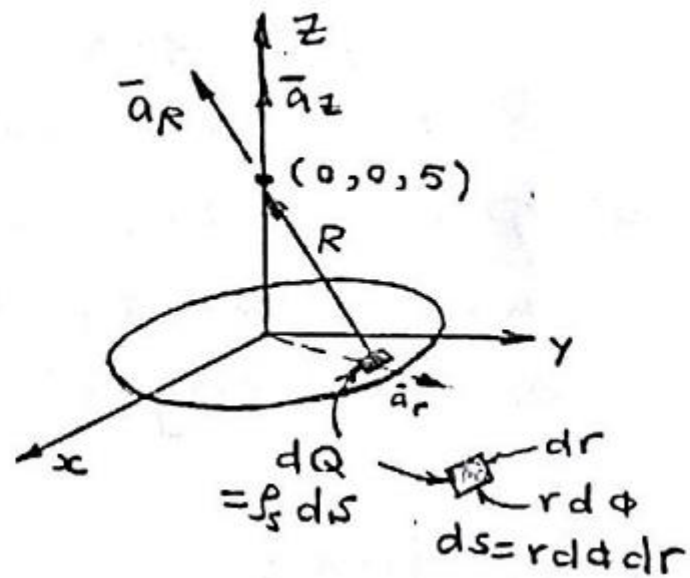
⇒ Solution :

As shown in fig.

we have :

$$\rho_s = \frac{Q}{A} = \frac{Q}{\pi r^2} = \frac{500\pi \times 10^{-6}}{\pi (5)^2}$$

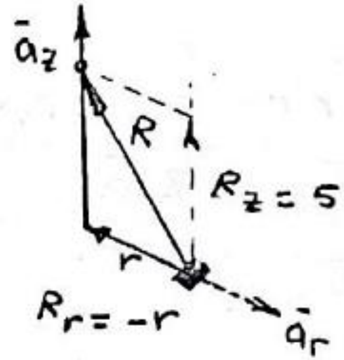
$$= 0.2 \times 10^{-4} \ C/m^2$$



In cylindrical coordinates (r, ϕ, z):

$$\bar{R} = -r\,\bar{a}_r + 5\,\bar{a}_z$$

$$\therefore\ R = \sqrt{r^2 + (5)^2}$$



$r = 0 \to 5$

$R = R_r\,\bar{a}_r + R_z\,\bar{a}_z$

Then each differential charge $(dQ = \rho_s\, ds)$ (as shown in fig 2-6) results in a differential force:

$$d\bar{F} = \frac{q\, dQ}{4\pi\epsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{(50\times10^{-6})(\rho_s\, r\, d\phi\, dr)}{4\pi(10^{-9}/36\pi)(r^2+25)}\;\frac{(-r\,\bar{a}_r + 5\,\bar{a}_z)}{\sqrt{r^2+25}}$$

$$\bar{F} = \int_{\phi=0}^{2\pi}\int_{r=0}^{5} \frac{(50\times10^{-6})(0.2\times10^{-4})\,5r\,dr\,d\phi}{4\pi(10^{-9}/36\pi)(r^2+25)^{3/2}}\,\bar{a}_z$$

$$= 45\times2\pi\int_0^5 \frac{r\,dr}{(r^2+25)^{3/2}} = 90\pi\left[\frac{-1}{\sqrt{r^2+25}}\right]_0^5 \bar{a}_z$$

$$= 16.56\,\bar{a}_z \quad N$$

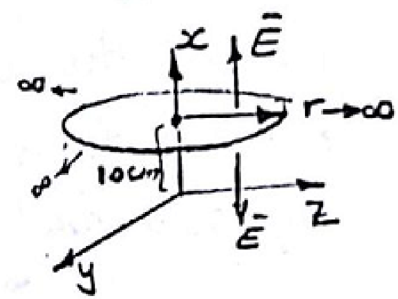
Note that

$$\Rightarrow \int \frac{x\,dx}{(ax^2+c)^{3/2}} = \frac{-1}{a\sqrt{ax^2+c}}$$

Ex (5) A circular disk of radius $r = \infty$ m is located at $\Rightarrow$ the plane $x = 10$ cm with a uniform charge density $\rho_s = (1/3\pi)$ nC/m². Determine the electric field intensity $\bar{E}$ at all points.

$\Rightarrow$ Solution :

(Note : the disk is considered an infinite sheet because $r = \infty$ m). Then from the above derivation of $\bar{E}$, we have that $\bar{E} = E_x \bar{a}_x + \cancel{E_r} \bar{a}_r$ ($\bar{E}_r = 0$ as seen before),



then:

$$|\bar{E}| = \frac{\rho_s}{2\epsilon_o} = \frac{(1/3\pi)\,10^{-9}}{2(10^{-9}/36\pi)} = 6 \text{ V/m}$$

Then: $\bar{E} = E\,\bar{a}_x$ (where the radial $\bar{E}_r = 0$)

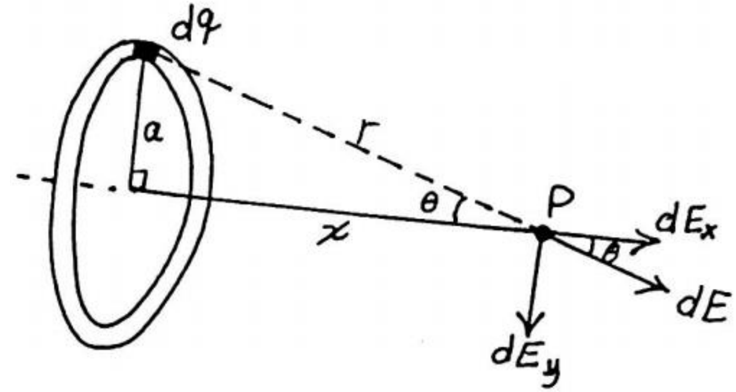
$\Rightarrow$ Above the sheet ($x > 10$ cm), $\bar{E} = 6\,\bar{a}_x$ V/m

and:

$\Rightarrow$ For the sheet when ($x < 10$ cm), $\bar{E} = -6\,\bar{a}_x$ V/m

Case (5)

For a uniform ring of charge Q and radius a, Calculate the electric field at Point P on the axis of the ring at distance x from the Centre ?



$$\because \quad dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$

$$\because \quad r = \sqrt{x^2+a^2}$$

$$\therefore \quad dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+a^2)} \longrightarrow (1)$$

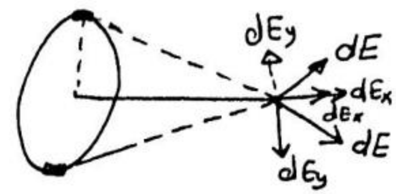
نحلل dE في اتجاهي x , y

$$dE_x = dE \cos\theta \qquad \cos\theta = \frac{x}{\sqrt{x^2+a^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+a^2)}\frac{x}{(x^2+a^2)^{1/2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{dq\, x}{(x^2+a^2)^{3/2}}$$

$$dE_y = dE \sin\theta$$



$$\therefore E_x = \int dE_x = \int \frac{1}{4\pi\varepsilon_0} \frac{dq\, x}{(x^2+a^2)^{3/2}}$$

$$E_x = \frac{1}{4\pi\varepsilon_0} \frac{q\, x}{(x^2+a^2)^{3/2}}$$

$$\therefore E = \frac{1}{4\pi\varepsilon_0} \frac{q\, x}{(x^2+a^2)^{3/2}}$$

باتجاه الـ x

* At the centre of the ring $x = 0$

$$\therefore E = 0$$

* If $x \gg a$ $\therefore E = \frac{1}{4\pi\varepsilon_0} \frac{q}{x^2}$

EX (6) : **Find E at the origin if the following charge distributions are present in free space: point charge, 12 nC at *P(2, 0, 6); uniform line charge density, 3nC/m at x = −2, y = 3; uniform surface charge density,0.2nC/m² at x = 2.***

Solution:

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$$\therefore \vec{E}_{origin} = \frac{Q}{4\pi\varepsilon_o R^2}\hat{a}_R + \frac{\rho_L}{2\pi\varepsilon_o \rho}\hat{a}_\rho + \frac{\rho_S}{2\varepsilon_o}\hat{a}_N$$

$$E = \left[\frac{12x10^{-9}}{4\pi\varepsilon_o}\frac{(-2a_x - 6a_z)}{(4+36)^{1.5}}\right] + \left[\frac{3x10^{-9}}{2\pi\varepsilon_o}\frac{(2a_x - 3a_y)}{(4+9)}\right] + \left[\frac{0.2x10^{-9}}{2\varepsilon_o}(-a_x)\right]$$

$$= -3.9a_x - 12.4a_y - 2.5a_z \quad V/m$$

Thank you for your attention

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